

REMARKS ON THE METRIC INDUCED BY THE ROBIN FUNCTION III

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ABSTRACT. Let D be a smoothly bounded pseudoconvex domain in \mathbf{C}^n , $n > 1$. Using the Robin function $\Lambda(p)$ that arises from the Green function $G(z, p)$ for D with pole at $p \in D$ associated with the standard sum-of-squares Laplacian, N. Levenberg and H. Yamaguchi had constructed a Kähler metric (the so-called Λ -metric) on D . In this article, we study the existence of geodesic spirals for this metric.

1. INTRODUCTION

We continue the study of the metric induced by the Robin function on strongly pseudoconvex domains in \mathbf{C}^n from [1] and [2]. To quickly recall the setup, for a smoothly bounded pseudoconvex domain $D \subset \mathbf{C}^n$, the Λ -metric on D is defined as

$$ds^2 = \sum_{\alpha, \beta=1}^n \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \otimes d\bar{z}_\beta$$

where $\Lambda(p) = \lim_{z \rightarrow p} (G(z, p) - |z - p|^{-2n+2})$ is the Robin function associated to the \mathbf{R}^{2n} -Green function $G(z, p)$ with pole at $p \in D$. It was proved in [7] that $\log(-\Lambda)$ is strictly plurisubharmonic and hence ds^2 defines a Kähler metric, which is however not invariant under biholomorphisms in general. Despite this seeming drawback, the Λ -metric on a strongly pseudoconvex domain $D \subset \mathbf{C}^n$ shares several properties with the Bergman metric (which is an invariant Kähler metric!). For example, it was shown in [2] that the Λ -metric on a strongly pseudoconvex domain D has the same boundary asymptotics as those of the Bergman metric (and hence the Kobayashi and also the Carathéodory metric) which implies that it is complete and that the metric space (D, ds^2) is Gromov hyperbolic. Also, the results of [1] show that the holomorphic sectional curvature of ds^2 along normal directions approaches $-1/(n-1)$ at the boundary, which is much like what is known for the Bergman metric. To carry this similarity further, it was shown in [6] that on a nonsimply connected strongly pseudoconvex domain D , every nontrivial homotopy class of closed loops in $\pi_1(D)$ contains a closed geodesic in the Bergman metric. It is also known that (see [3], [4], [8]) for a smooth strongly pseudoconvex domain D , the space of harmonic forms $\mathcal{H}^{p,q}(D)$ with respect to the Bergman metric is zero dimensional if $p+q \neq n$ while it is infinite dimensional for $p+q = n$. Using the fact that the boundary asymptotics of the Bergman metrics match those of the Λ -metric, the exact analogues of both results were shown to hold for the Λ -metric as well in [1].

The purpose of this note is to identify one more property that is shared by the Λ -metric and the Bergman metric thus increasing the list of their similarities by one. We first need a definition. Let (M, g) be a complete Riemannian manifold. A *geodesic spiral* is a geodesic $c : \mathbb{R} \rightarrow M$ such that there is a compact subset $K \subset M$ with $c(t) \in K$ for all $t \geq 0$ and c is not closed. Further, if $c : \mathbf{R} \rightarrow M$ is a non-constant geodesic and there exist times $t_1, t_2 \in \mathbf{R}$ with $t_1 < t_2$ such that $c(t_1) = c(t_2)$, then the curve $c(t)$ restricted to the interval $[t_1, t_2]$ will be called a *geodesic loop* through the point $c(t_1) = c(t_2) \in M$.

Theorem 1.1. *Let D be a smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n and suppose that the universal cover of D is infinitely sheeted. Then for each $p_0 \in D$ which does not lie on a closed geodesic there exists a geodesic spiral for the Λ -metric passing through p_0 .*

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The analogous result for the Bergman metric on smoothly bounded strongly pseudoconvex domains can be found in [6]. The main step is Lemma 2.2 of [6] which states that if (M, g) is a complete Riemannian manifold whose universal cover is infinitely sheeted and $x_0 \in M$ is a point through which no closed geodesic passes and $K \subset M$ is a compact set which contains all possible geodesic loops through x_0 , then there is a geodesic spiral passing through x_0 . By appealing to this, the theorem follows if we can show that there exists a compact set $K \subset D$ that contains all the possible geodesic spirals through p_0 . Thus the problem reduces to finding such a compact K . To do this, let ψ be a globally smooth defining function for the strongly pseudoconvex domain D .

Proposition 1.2. *There exists an $\epsilon = \epsilon(D) > 0$ such that for each geodesic $\gamma : \mathbf{R} \rightarrow D$ for the Λ -metric with $\psi(\gamma(0)) > -\epsilon$ and $(\psi \circ \gamma)'(0) = 0$, it follows that $(\psi \circ \gamma)''(0) > 0$.*

Take this $\epsilon > 0$ and let $2\epsilon_1 = \min\{\epsilon, \psi(p_0)\}$. Then

$$K = \{p \in D : \psi(p) \leq -\epsilon_1\}$$

is the compact set that we are seeking. Indeed, let $\gamma : [t_1, t_2] \rightarrow D$ be a geodesic loop with $p_0 = \gamma(t_1) = \gamma(t_2)$. Suppose that γ does not lie in K , i.e., γ enters the ϵ_1 band around the boundary ∂D . But then, being a loop, it must turn back and hence $\psi \circ \gamma$ must have a maximum somewhere, sat at $t_0 \in (t_1, t_2)$. This implies that $(\psi \circ \gamma)(t_0) > -\epsilon$, $(\psi \circ \gamma)'(t_0) = 0$ and $(\psi \circ \gamma)''(t_0) < 0$ which contradicts the proposition. Thus it suffices to prove Proposition 1.2.

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2. ASYMPTOTICS OF Λ AND λ

We begin by strengthening some of the boundary asymptotics of the Λ -metric from [2]. Let D be a C^∞ -smoothly bounded domain in \mathbf{C}^n with a C^∞ -smooth defining function ψ . In what follows, the standard convention of denoting derivatives by suitable subscripts will be followed. For example, $\psi_\alpha = \partial\psi/\partial p_\alpha$, $\psi_{\alpha\bar{\beta}} = \partial^2\psi/\partial p_\alpha\partial\bar{p}_\beta$, etc. Also, let $\partial\psi = (\psi_1, \dots, \psi_n)$. The *normalised* Robin function λ associated to (D, ψ) is defined by

$$\lambda(p) = \begin{cases} \Lambda(p)\psi(p)^{2n-2} & \text{if } p \in D, \\ -|\partial\psi(p)|^{2n-2} & \text{if } p \in \partial D. \end{cases}$$

This function has the following geometric significance: For $p \in D$, let $D(p)$ be the domain in \mathbf{C}^n obtained by applying the affine transformation $z \mapsto (z - p)/(-\psi(p))$ to D , i.e.,

$$D(p) = \left\{ w \in \mathbf{C}^n : w = \frac{z - p}{-\psi(p)} \right\}.$$

Observe that $D(p)$ contains the origin and by [9, Prop. 5.1],

$$\Lambda_{D(p)}(0) = \Lambda(p)\psi(p)^{2n-2} = \lambda(p).$$

Also, for $p \in \partial D$, let $D(p)$ be the half-space defined by

$$D(p) = \left\{ w \in \mathbf{C}^n : 2\Re\left(\sum_{\alpha=1}^n \psi_\alpha(p)w_\alpha\right) - 1 < 0 \right\}.$$

Again, $D(p)$ contains the origin and by [2, (1.4)],

$$\Lambda_{D(p)}(0) = -|\partial\psi(p)|^{2n-2} = \lambda(p).$$

Thus $\lambda(p)$ is the Robin constant for $D(p)$ at the origin. In [7], this geometric significance of λ was used to understand its regularity near the boundary ∂D . Indeed, let

$$\mathcal{D} = \cup_{p \in D \cup \partial D} (p, D(p)) = \{(p, w) : p \in D, w \in D(p)\}.$$

The $\mathcal{D} : p \mapsto D(p)$ is a smooth variation of domains in \mathbf{C}^n defined by the smooth function on $\mathbf{C}^n \times \mathbf{C}^n$,

$$(2.1) \quad f(p, w) = 2\Re \left\{ \sum_{\alpha=1}^n \int_0^1 (w_\alpha \psi_\alpha(p - \psi(p)tw)) dt \right\} - 1.$$

Suppose $g(p, w)$ is the Green function for $D(p)$ with pole at the origin. Then we have the first variation formula

$$(2.2) \quad \frac{\partial g}{\partial p_\alpha}(p, w) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p)} k_1^{(\alpha)}(p, \zeta) |\partial_\zeta g(p, \zeta)| \frac{\partial g_w}{\partial n_\zeta}(p, \zeta) dS_\zeta, \quad p \in D, w \in D(p).$$

Here, σ_{2n} is the surface area of the unit sphere in \mathbf{R}^{2n} , dS_ζ is the surface area element on $\partial D(p)$, $\partial_\zeta g = (\partial g / \partial \zeta_1, \dots, \partial g / \partial \zeta_n)$, $g_w(p, w)$ is the Green function for $D(p)$ with pole at w , n_ζ is the unit outward normal to $\partial D(p)$ at ζ , and

$$(2.3) \quad k_1^{(\alpha)}(p, \zeta) = \frac{\partial f}{\partial p_\alpha}(p, \zeta) / |\partial_\zeta f(p, \zeta)|.$$

When $p \in D$ converges to $p_0 \in \partial D$ and $w \in D(p)$ converges to $w_0 \in D(p_0)$, then the integral (2.2) converges to

$$(2.4) \quad \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p_0)} k_1^{(\alpha)}(p_0, \zeta) |\partial_\zeta g(p_0, \zeta)| \frac{\partial g_{w_0}}{\partial n_\zeta}(p_0, \zeta) dS_\zeta.$$

Then using a standard argument (see Step 6, Chapter 3 of [7]) it was shown that $\partial g / \partial p_\alpha(p_0, w_0)$ exists and is equal to the above integral. It follows that $g(p, w)$ is a C^1 -smooth function of p up to ∂D and (2.2) holds for $p \in \partial D$ also. From [9, (1.3)], λ is also a C^1 -smooth function of p up to ∂D . Also, since

$$\lambda_\alpha(p) = \frac{\partial g}{\partial p_\alpha}(p, 0), \quad p \in D,$$

we note that for all $p \in \overline{D}$,

$$(2.5) \quad \lambda_\alpha(p) = -\frac{1}{(n-1)\sigma_{2n}} \int_{\partial D(p)} k_1^{(\alpha)}(p, \zeta) |\partial_\zeta g(p, \zeta)|^2 dS_\zeta.$$

Similarly, using the second variation formula it was shown that $g(p, w)$ and thus $\lambda(p)$ is a C^2 -smooth function of p up to ∂D .

In [1], we studied the boundary behaviour of Λ and λ under a C^∞ -perturbation of D . In this section, we derive some consequences of these results that will be used to prove the main theorem. First, let D_ν , $\nu \geq 1$, be C^∞ -smoothly bounded domains in \mathbf{C}^n with C^∞ -smooth defining functions ψ_ν , such that $\{\psi_\nu\}$ converges in the C^∞ -topology on compact subsets of \mathbf{C}^n to ψ . The normalised Robin function associated to (D_ν, ψ_ν) will be denoted by λ_ν . For multi-indices $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $B = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbf{N}$, let

$$D^A = \frac{\partial^{|A|}}{\partial p_1^{\alpha_1} \partial p_2^{\alpha_2} \dots \partial p_n^{\alpha_n}} \quad \text{and} \quad D^{\overline{B}} = \frac{\partial^{|B|}}{\partial \overline{p}_1^{\beta_1} \partial \overline{p}_2^{\beta_2} \dots \partial \overline{p}_n^{\beta_n}}$$

and let $D^{A\overline{B}} = D^A D^{\overline{B}}$. We have from [1]:

Theorem 2.1. *Suppose $p_\nu \in D_\nu$ converges to $p_0 \in \partial D$. Define the half space*

$$\mathcal{H} = \left\{ w \in \mathbf{C}^n : 2\Re \left(\sum_{\alpha=1}^n \psi_\alpha(p_0) w_\alpha \right) - 1 < 0 \right\}$$

and let $\Lambda_{\mathcal{H}}$ denote the Robin function for \mathcal{H} . Then for all multi-indices $A, B \in \mathbf{N}$,

$$(-1)^{|A|+|B|} D^{A\overline{B}} \Lambda_\nu(p_\nu) (\psi_\nu(p_\nu))^{2n-2+|A|+|B|} \rightarrow D^{A\overline{B}} \Lambda_{\mathcal{H}}(0)$$

as $\nu \rightarrow \infty$.

For the half space \mathcal{H} , we have the explicit formula

$$(2.6) \quad G_{\mathcal{H}}(p, z) = |z - p|^{-2n+2} - |z - p^*|^{-2n+2},$$

where p^* is the symmetric point of p given by

$$(2.7) \quad p^* = p - \left(\frac{2\Re(\sum_{\alpha=1}^n \psi_{\alpha}(p_0)p_{\alpha}) - 1}{|\partial\psi(p_0)|^2} \right) \bar{\partial}\psi(p_0).$$

Therefore, the Robin function for \mathcal{H} is

$$\Lambda_{\mathcal{H}}(p) = -|p - p^*|^{-2n+2} = -|\partial\psi(p_0)|^{2n-2} \left\{ 2\Re\left(\sum_{\alpha=1}^n \psi_{\alpha}(p_0)p_{\alpha}\right) - 1 \right\}^{-2n+2}.$$

Thus we can compute $D^{A\bar{B}}\Lambda_{\mathcal{H}}(0)$ explicitly for all multi-indices A, B , and hence the above theorem provides the boundary asymptotics of all derivatives of Λ_{ν} . For our record, we now write down few of them in the corollary that follows. Let

$$\mathbf{I} = \{1, \dots, n\} \quad \text{and} \quad \bar{\mathbf{I}} = \{\bar{1}, \dots, \bar{n}\}.$$

If $a \in \bar{\mathbf{I}}$, then let $p_a = \bar{p}_{\bar{a}}$.

Corollary 2.2. *Under the hypothesis of Theorem 2.1, we have for all $a, b, c \in \mathbf{I} \cup \bar{\mathbf{I}}$,*

- (a) $\lim_{\nu \rightarrow \infty} \Lambda_{\nu}(p_{\nu})(\psi_{\nu}(p_{\nu}))^{2n-2} = -|\partial\psi(p_0)|^{2n-2},$
- (b) $\lim_{\nu \rightarrow \infty} \frac{\partial \Lambda_{\nu}}{\partial p_a}(p_{\nu})(\psi_{\nu}(p_{\nu}))^{2n-1} = (2n-2)\psi_a(p_0)|\partial\psi(p_0)|^{2n-2},$
- (c) $\lim_{\nu \rightarrow \infty} \frac{\partial^2 \Lambda_{\nu}}{\partial p_a \partial p_b}(p_{\nu})(\psi_{\nu}(p_{\nu}))^{2n} = -(2n-2)(2n-1)\psi_a(p_0)\psi_b(p_0)|\partial\psi(p_0)|^{2n-2},$ and
- (d) $\lim_{\nu \rightarrow \infty} \frac{\partial^3 \Lambda_{\nu}}{\partial p_a \partial p_b \partial p_c}(p_{\nu})(\psi_{\nu}(p_{\nu}))^{2n+1} = (2n-2)(2n-1)2n\psi_a(p_0)\psi_b(p_0)\psi_c(p_0)|\partial\psi(p_0)|^{2n-2}.$

Now let $g_{\alpha\bar{\beta}}$ and $g_{\nu\alpha\bar{\beta}}$, $1 \leq \alpha, \beta \leq n$, be the components of the Λ -metric on D and D_{ν} respectively. Note that

$$(2.8) \quad g_{\alpha\bar{\beta}} = \frac{\partial^2 \log(-\Lambda)}{\partial p_{\alpha} \partial \bar{p}_{\beta}} = \frac{\Lambda_{\alpha\bar{\beta}}}{\Lambda} - \frac{\Lambda_{\alpha}\Lambda_{\bar{\beta}}}{\Lambda^2},$$

and by differentiating this with respect to p_{γ} , $1 \leq \gamma \leq n$,

$$(2.9) \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial p_{\gamma}} = \frac{\Lambda_{\alpha\bar{\beta}\gamma}}{\Lambda} - \left(\frac{\Lambda_{\alpha\bar{\beta}}\Lambda_{\gamma}}{\Lambda^2} + \frac{\Lambda_{\alpha\gamma}\Lambda_{\bar{\beta}}}{\Lambda^2} + \frac{\Lambda_{\bar{\beta}\gamma}\Lambda_{\alpha}}{\Lambda^2} \right) + \frac{2\Lambda_{\alpha}\Lambda_{\bar{\beta}}\Lambda_{\gamma}}{\Lambda^3}.$$

Multiplying the corresponding equations for $g_{\nu\alpha\bar{\beta}}$ by ψ_{ν}^2 and ψ_{ν}^3 respectively we obtain from Corollary 2.2 that

Corollary 2.3. *Under the hypothesis of Theorem 2.1, we have for all $\alpha, \beta, \gamma \in \mathbf{I}$,*

- (a) $\lim_{\nu \rightarrow \infty} g_{\nu\alpha\bar{\beta}}(p_{\nu})(\psi_{\nu}(p_{\nu}))^2 = (2n-2)\psi_{\alpha}(p_0)\psi_{\bar{\beta}}(p_0),$ and
- (b) $\lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu\alpha\bar{\beta}}}{\partial p_{\gamma}}(p_{\nu})(\psi_{\nu}(p_{\nu}))^3 = -2(2n-2)\psi_{\alpha}(p_0)\psi_{\bar{\beta}}(p_0)\psi_{\gamma}(p_0).$

In the proof of the main theorem, we will be particularly interested in these asymptotics when D and D_{ν} , $\nu \geq 1$, are in the following form:

- (†) $\left\{ \begin{array}{l} \bullet D \text{ is strongly pseudoconvex, } 0 \in \partial D, \text{ and } \partial\psi(0) = (0, \dots, 0, 1), \\ \bullet D_{\nu} \text{ is strongly pseudoconvex, } 0 \in \partial D_{\nu}, \text{ and } \partial\psi_{\nu}(0) = (0, \dots, 0, 1), \text{ and} \\ \bullet p_{\nu} \text{ lies on the inner normal to } \partial D_{\nu} \text{ and } p_{\nu} \rightarrow 0. \end{array} \right.$

Under this normalisation, observe in corollary 2.2 that if any of the derivatives is with respect to a variable other than p_n or \bar{p}_n , then the asymptotics become 0. This means that these are not the strongest asymptotics in this case. Similarly, the asymptotics in corollary 2.3 are not

the strongest one unless $\alpha = \beta = \gamma = n$. The problem with these weak asymptotics is that they make $\det(g_{\alpha\bar{\beta}})$ indeterminate. Indeed,

$$g_{\alpha\bar{\beta}}(p) \sim \frac{\psi_\alpha(0)\psi_{\bar{\beta}}(0)}{(\psi(p))^2},$$

for p close to 0 and hence

$$\det(g_{\alpha\bar{\beta}}(p)) \sim \frac{\det(\psi_\alpha(0)\psi_{\bar{\beta}}(0))}{(\psi(p))^{2n}},$$

which is apriori indeterminate since the numerator vanishes and $\psi(p) \rightarrow 0$ as $p \rightarrow 0$. Thus it is necessary to improve these asymptotics for the calculation of geodesics. This can be done for the first and the second order derivatives of Λ_ν and for $g_{\nu\alpha\bar{\beta}}$, by means of the following theorem from [1]:

Theorem 2.4. *Suppose $p_\nu \in D_\nu$ converges to $p_0 \in \partial D$. Then for all $\alpha, \beta \in \mathbf{I}$,*

- (a) $\lim_{\nu \rightarrow \infty} \lambda_\nu(p_\nu) = \lambda(p_0)$,
- (b) $\lim_{\nu \rightarrow \infty} \frac{\partial \lambda_\nu}{\partial p_\alpha}(p_\nu) = \lambda_\alpha(p_0)$,
- (c) $\lim_{\nu \rightarrow \infty} \frac{\partial^2 \lambda_\nu}{\partial p_\alpha \partial \bar{p}_\beta}(p_\nu) = \lambda_{\alpha\bar{\beta}}(p_0)$.

To see how this theorem leads to finer asymptotics, differentiate the normalised Robin function

$$\lambda = \Lambda \psi^{2n-2},$$

with respect to a , and then with respect to b , to obtain

$$(2.10) \quad \Lambda_a \psi^{2n-2} = \lambda_a - (2n-2)\lambda \psi^{-1} \psi_a,$$

and

$$(2.11) \quad \Lambda_{ab} \psi^{2n-1} = \lambda_{ab} \psi - (2n-2)(\lambda_a \psi_b + \lambda_b \psi_a) + (2n-2)(2n-1)\lambda \psi^{-1} \psi_a \psi_b - (2n-2)\lambda \psi_{ab}.$$

While Theorem 2.4 provides information about the derivatives of λ_ν in the above formulae corresponding to Λ_ν , the terms of the form $\psi_\nu^{-1}(\partial \psi_\nu / \partial p_a)$, can be controlled by the following:

Lemma 2.5. *Under the normalisation (\dagger) , we have for all $a \in \mathbf{I} \cup \bar{\mathbf{I}}$, $a \neq n, \bar{n}$,*

$$\lim_{\nu \rightarrow \infty} \frac{1}{\psi_\nu(p_\nu)} \frac{\partial \psi_\nu}{\partial p_a}(p_\nu) = \frac{1}{2} (\psi_{an}(0) + \psi_{a\bar{n}}(0)).$$

For a proof, see [1, Lemma 6.2]. We also need to compute $\lambda_a(0)$.

Lemma 2.6. *Under the normalisation (\dagger) , we have for all $a \in \mathbf{I} \cup \bar{\mathbf{I}}$, $a \neq n, \bar{n}$,*

$$\lambda_a(0) = -(n-1)(\psi_{an}(0) + \psi_{a\bar{n}}(0))$$

Proof. Let

$$\mathcal{H} = D(0) = \{w \in \mathbf{C}^n : 2\Re w_n - 1 < 0\}.$$

From (2.5) we have

$$\lambda_a(0) = \frac{\partial g}{\partial p_a}(0, 0) = -\frac{1}{(n-1)\sigma_{2n}} \int_{\partial \mathcal{H}} k_1^{(a)}(0, \zeta) |\partial_\zeta g(0, \zeta)|^2 dS_\zeta.$$

Let us first compute $k_1^{(a)}(0, \zeta)$ from (2.3). Differentiating (2.1) with respect to p_a and using $\psi_a(0) = 0$, we obtain

$$\frac{\partial f}{\partial p_a}(0, \zeta) = \sum_{j=1}^n (\zeta_j \psi_{aj}(0) + \bar{\zeta}_j \psi_{a\bar{j}}(0)).$$

Also,

$$\frac{\partial f}{\partial \zeta_\alpha}(0, \zeta) = \psi_\alpha(0)$$

so that $|\partial_\zeta f(0, \zeta)| = 1$. Thus

$$k_1^{(a)}(0, \zeta) = \frac{\partial f}{\partial p_a}(0, \zeta) / |\partial_\zeta f(0, \zeta)| = \sum_{j=1}^n (\zeta_j \psi_{aj}(0) + \bar{\zeta}_j \psi_{a\bar{j}}(0)).$$

From (2.6),

$$g(0, \zeta) = |\zeta|^{-2n+2} - |\zeta - 0^*|^{-2n+2}$$

where $0^* = (0, \dots, 0, 1)$ is the symmetric point of 0 with respect to the hyperplane $\partial\mathcal{H}$. Therefore,

$$\frac{\partial g}{\partial \zeta_j}(0, \zeta) = -(n-1)(\bar{\zeta}_j |\zeta|^{-2n} - (\bar{\zeta}_j - \bar{0}_j^*) |\zeta - 0^*|^{-2n}), \quad 1 \leq j \leq n.$$

Note that for $\zeta \in \partial\mathcal{H}$, $|\zeta| = |\zeta - 0^*|$. Therefore,

$$\frac{\partial g}{\partial \zeta_i}(0, \zeta) = -(n-1)|\zeta|^{-2n} \bar{0}_i^*, \quad \zeta \in \partial\mathcal{H}, 1 \leq j \leq n.$$

This implies that

$$|\partial_\zeta g(0, \zeta)| = (n-1)|\zeta|^{-2n}, \quad \zeta \in \partial\mathcal{H}.$$

Thus

$$(2.12) \quad \lambda_a(0) = -\frac{(n-1)}{\sigma_{2n}} \sum_{j=1}^n \left\{ \psi_{aj}(0) \int_{\partial\mathcal{H}} \zeta_j |\zeta|^{-4n} dS_\zeta + \psi_{a\bar{j}}(0) \int_{\partial\mathcal{H}} \bar{\zeta}_j |\zeta|^{-4n} dS_\zeta \right\}.$$

Let us now compute the above integrals. First observe that for $1 \leq j \leq n-1$,

$$\frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} \zeta_j |\zeta|^{-4n} dS_\zeta = \frac{1}{\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{x_j + iy_j}{(x_1^2 + y_1^2 + \cdots + 1/4 + y_n^2)^{2n}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n = 0,$$

where as usual $\widehat{dx_n}$ means that the surface measure dS_ζ does not contain the covector dx_n . Also,

$$\begin{aligned} \frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} \zeta_n |\zeta|^{-4n} dS_\zeta &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1/2 + iy_n}{(x_1^2 + y_1^2 + \cdots + 1/4 + y_n^2)^{2n}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n \\ &= \frac{1}{2\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(x_1^2 + y_1^2 + \cdots + 1/4 + y_n^2)^{2n}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n \equiv \frac{1}{2} X. \end{aligned}$$

Using polar coordinates,

$$X = \frac{\sigma_{2n-1}}{\sigma_{2n}} \int_0^\infty \frac{r^{2n-2}}{(r^2 + 1/4)^{2n}} dr \equiv I(2n-2, 2n).$$

Repeated integration by parts yields

$$I(2n-2, 2n) = \frac{2n-3}{2(2n-1)} \frac{2n-5}{2(2n-2)} \cdots \frac{1}{2(n+1)} I(0, n+1).$$

By the residue theorem,

$$I(0, n+1) = \frac{\pi}{n!} (n+1)(n+2) \cdots (2n).$$

Also, since $\sigma_m = 2\pi^{m/2}/\Gamma(m/2)$,

$$\frac{\sigma_{2n-1}}{\sigma_{2n}} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n)}{\Gamma(n-1/2)} = \frac{1}{\pi} \frac{2^{n-1}(n-1)!}{(2n-3)(2n-5) \cdots 1}.$$

Thus,

$$X = \left\{ \frac{1}{\pi} \frac{2^{n-1}(n-1)!}{(2n-3)(2n-5) \cdots 1} \right\} \left\{ \frac{2n-3}{2(2n-1)} \frac{2n-5}{2(2n-2)} \cdots \frac{1}{2(n+1)} \right\} \left\{ \frac{\pi}{n!} (n+1)(n+2) \cdots (2n) \right\} = 2,$$

and hence

$$\frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} \zeta_n |\zeta|^{-4n} dS_\zeta = 1.$$

It follows from (2.12) that

$$\lambda_a(0) = -(n-1)(\psi_{an}(0) + \psi_{a\bar{n}}(0)).$$

□

Corollary 2.7. *Under the normalisation (\dagger) , we have for all $a, b \in \mathbf{I} \cup \bar{\mathbf{I}}$, $a \neq n, \bar{n}$,*

- (a) $\lim_{\nu \rightarrow \infty} \frac{\partial \Lambda_\nu}{\partial p_a}(p_\nu)(\psi_\nu(p_\nu))^{2n-2} = 0,$
- (b) $\lim_{\nu \rightarrow \infty} \frac{\partial^2 \Lambda_\nu}{\partial p_a \partial p_b}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} = -(n-1)\psi_b(0)(\psi_{an}(0) + \psi_{a\bar{n}}(0)) + (2n-2)\psi_{ab}(0).$

Proof. Applying Theorem 2.4 and Lemma 2.5 to (2.10) corresponding to Λ_ν ,

$$\lim_{\nu \rightarrow \infty} \frac{\partial \Lambda_\nu}{\partial p_a}(p_\nu)\psi(p_\nu)^{2n-2} = \lambda_a(0) - (n-1)\lambda(0)(\psi_{an}(0) + \psi_{a\bar{n}}(0)) = 0,$$

in view of Lemma 2.6 and the fact that $\lambda(0) = -|\partial\psi(0)|^{2n-2} = -1$. Hence (a) is proved.

Applying Theorem 2.4 and Lemma 2.5 to (2.11) corresponding to Λ_ν , we obtain

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \frac{\partial^2 \Lambda_\nu}{\partial p_a \partial p_b}(p_\nu)\psi_\nu(p_\nu)^{2n-1} &= -(2n-2)\lambda_a(0)\psi_b(0) + (n-1)(2n-1)\lambda(0)\psi_b(0)\{\psi_{an}(0) + \psi_{a\bar{n}}(0)\} \\ &\quad - (2n-2)\lambda(0)\psi_{ab}(0) = -(n-1)\psi_b(0)\{\psi_{an}(0) + \psi_{a\bar{n}}(0)\} + (2n-2)\psi_{ab}(0), \end{aligned}$$

in view of Lemma 2.6 and the fact that $\lambda(0) = -1$. Hence (b) is proved. □

Now, multiplying (2.8) by ψ , we may write

$$g_{\alpha\bar{\beta}}\psi = \frac{\Lambda_{\alpha\bar{\beta}}\psi^{2n-1}}{\Lambda\psi^{2n-2}} - \frac{(\Lambda_\alpha\psi^{2n-2})(\Lambda_{\bar{\beta}}\psi^{2n-1})}{(\Lambda\psi^{2n-2})^2}.$$

Applying Corollary 2.7 to the above formula corresponding to g_ν , we obtain the following:

Corollary 2.8. *Under the normalisation (\dagger) , we have for all $\alpha, \beta \in \mathbf{I}$, $\alpha \neq n$,*

$$g_{\nu\alpha\bar{\beta}}(p_\nu)\psi_\nu(p_\nu) = (n-1)\psi_{\bar{\beta}}(0)(\psi_{\alpha n}(0) + \psi_{\alpha\bar{n}}(0)) - (2n-2)\psi_{\alpha\bar{\beta}}(0).$$

These asymptotics are strong enough to controll $\det(g_{\alpha\bar{\beta}})$. Indeed,

Corollary 2.9. *Under the normalisation (\dagger) , we have*

$$(2.13) \quad \lim_{\nu \rightarrow \infty} \det(g_{\nu\alpha\bar{\beta}}(p_\nu))\psi_\nu(p_\nu)^{n+1} = (-1)^{n-1}(2n-2)^n \det(\psi_{\alpha\bar{\beta}}(0))_{1 \leq \alpha, \beta \leq n-1},$$

which is nonzero as D is strongly pseudoconvex.

Proof. Let $(\Delta_{\alpha\bar{\beta}})$ be the cofactor matrix of $(g_{\alpha\bar{\beta}})$. Then

$$(2.14) \quad \det(g_{\alpha\bar{\beta}}) = g_{n\bar{1}}\Delta_{n\bar{1}} + \dots + g_{n\bar{n}}\Delta_{n\bar{n}}.$$

Note that

$$\begin{aligned} \Delta_{n\bar{\beta}}\psi^{n-1} &= \psi^{n-1}(-1)^{n+\beta} \det \begin{pmatrix} g_{1\bar{1}} & \dots & g_{1\bar{\beta-1}} & g_{1\bar{\beta+1}} & \dots & g_{1\bar{n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-1\bar{1}} & \dots & g_{n-1\bar{\beta-1}} & g_{n-1\bar{\beta+1}} & \dots & g_{n-1\bar{n}} \end{pmatrix} \\ &= (-1)^{n+\beta} \det \begin{pmatrix} g_{1\bar{1}}\psi & \dots & g_{1\bar{\beta-1}}\psi & g_{1\bar{\beta+1}}\psi & \dots & g_{1\bar{n}}\psi \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-1\bar{1}}\psi & \dots & g_{n-1\bar{\beta-1}}\psi & g_{n-1\bar{\beta+1}}\psi & \dots & g_{n-1\bar{n}}\psi \end{pmatrix}. \end{aligned}$$

Applying Corollary 2.8 to the above formula corresponding to g_ν , we observe that

$$\lim_{\nu \rightarrow \infty} \Delta_{\nu n\bar{\beta}}(p_\nu)(\psi_\nu(p_\nu))^{n-1}$$

exists and is finite. In addition, using $\psi_j(0) = 0$ for $1 \leq j < n$,

$$\lim_{\nu \rightarrow \infty} \Delta_{\nu n\bar{n}}\psi^{n-1} = (-1)^{n-1}(2n-2)^{n-1} \det(\psi_{\alpha\bar{\beta}}(0))_{1 \leq \alpha, \beta \leq n-1}.$$

Multiplying (2.14) by ψ^{n+1} , we may write

$$\det(g_{\alpha\bar{\beta}})\psi^{n+1} = (g_{n\bar{1}}\psi^2)(\Delta_{n\bar{1}}\psi^{n-1}) + \dots + (g_{n\bar{n}}\psi^2)(\Delta_{n\bar{n}}\psi^{n-1}).$$

Now applying the above asymptotics of the cofactors and Corollary 2.3 to this formula corresponding to g_ν , we obtain (2.13). \square

Corollary 2.10. *Under the normalisation (\dagger) ,*

$$\lim_{\nu \rightarrow \infty} \frac{1}{\psi_\nu(p_\nu)^2} g^{n\bar{\beta}}(p_\nu)$$

exists and is finite for all $\beta \in \mathbf{I}$, and in particular,

$$\lim_{\nu \rightarrow \infty} \frac{1}{\psi_\nu(p_\nu)^2} g^{n\bar{n}}(p_\nu) = \frac{1}{2n-2}.$$

Proof. Dividing

$$g^{n\bar{\beta}} = \frac{\Delta_{n\bar{\beta}}}{\det(g_{i\bar{j}})}$$

by ψ^2 , we may write

$$\frac{1}{\psi^2} g^{n\bar{\beta}} = \frac{1}{\det(g_{i\bar{j}}) \psi^{n+1}} \Delta_{n\bar{\beta}} \psi^{n-1}.$$

Now applying Corollary 2.9 and the asymptotics of the cofactors in its proof to the above formula corresponding to g_ν we obtain the desired results. \square

It is not known whether λ is C^3 -smooth up to ∂D and so the above procedure can not be applied to obtain finer asymptotics of the third order derivatives of Λ and thus of the derivatives of the metric components. However in [7, Chap. 6], a relation between the third order derivatives of Λ and certain derivatives of $g(p, w)$ was established which can be used to obtain information about finer asymptotics. Indeed, recall that

$$g(p, w) = (\psi(p))^{2n-2} G(p, z),$$

where $p, z \in D$ and $w = (z - p)/(-\psi(p))$. Differentiating the above equation with respect to z_α and with respect to p_α , away from the diagonal $z = p$, we obtain

$$\frac{1}{-\psi} \frac{\partial g}{\partial w_\alpha} = \psi^{2n-2} \frac{\partial G}{\partial z_\alpha},$$

and

$$\frac{\partial g}{\partial p_\alpha} + \frac{1}{\psi} \frac{\partial g}{\partial w_\alpha} + \frac{1}{\psi^2} \psi_\alpha \sum_{i=1}^n \left\{ (z_i - p_i) \frac{\partial g}{\partial w_i} + (\bar{z}_i - \bar{p}_i) \frac{\partial g}{\partial \bar{w}_i} \right\} = (2n-2) \psi^{2n-3} \psi_\alpha G + \psi^{2n-2} \frac{\partial G}{\partial p_\alpha}.$$

Adding these two equations and using $w_i = (z_i - p_i)/(-(\psi(p)))$, $g = \psi^{2n-2} G$, we obtain

$$(2.15) \quad \frac{\partial g}{\partial p_\alpha} - (2n-2) \frac{1}{\psi} \psi_\alpha g - \frac{1}{\psi} \psi_\alpha \sum_{i=1}^n \left(w_i \frac{\partial g}{\partial w_i} + \bar{w}_i \frac{\partial g}{\partial \bar{w}_i} \right) = \psi^{2n-2} \left(\frac{\partial G}{\partial z_\alpha} + \frac{\partial G}{\partial p_\alpha} \right)$$

away from the diagonal $z = p$. Now set

$$G_\alpha(p, z) = \left(\frac{\partial G}{\partial p_\alpha} + \frac{\partial G}{\partial z_\alpha} \right) (p, z) \quad \text{for } (p, z) \in D \times D, \alpha = 1, \dots, n,$$

which is, by [7, Prop. 6.1], a real analytic, symmetric function in $D \times D$, harmonic in z and p and satisfy

$$(2.16) \quad G_\alpha(p, p) = \frac{\partial \Lambda}{\partial p_\alpha}(p).$$

Also set

$$(2.17) \quad g_\alpha(p, w) = \psi(p) \frac{\partial g}{\partial p_\alpha}(p, w) - \psi_\alpha(p) \left\{ (2n-2)g(p, w) + \sum_{i=1}^n \left(w_i \frac{\partial g}{\partial w_i}(p, w) + \bar{w}_i \frac{\partial g}{\partial \bar{w}_i}(p, w) \right) \right\}$$

$p \in \bar{D}, w \in D(p), 1 \leq \alpha \leq n.$

which is, by [7, Prop. 6.2], a harmonic function of $w \in D(p)$ for each $p \in \overline{D}$, and satisfies

$$(2.18) \quad g_\alpha(p, 0) = \psi(p)\lambda_\alpha(p) - (2n-2)\psi_\alpha(p)\lambda(p) = \Lambda_\alpha(p).$$

Now (2.15) can be written as

$$g_\alpha(p, w) = (\psi(p))^{2n-1} G_\alpha(p, z)$$

for $p, z \in D$ and $w = (z - p)/(-\psi(p))$. Repeating the above calculation for this relation, we obtain

$$(2.19) \quad \frac{\partial g_\alpha}{\partial \overline{p}_\beta} - (2n-1)\frac{1}{\psi}\psi_\beta g_\alpha - \frac{1}{\psi}\psi_\beta \sum_{i=1}^n \left(w_i \frac{\partial g_\alpha}{\partial w_i} + \overline{w}_i \frac{\partial g_\alpha}{\partial \overline{w}_i} \right) = \psi^{2n-1} \left(\frac{\partial G_\alpha}{\partial \overline{z}_\beta} + \frac{\partial G_\alpha}{\partial \overline{p}_\beta} \right).$$

Again set

$$G_{\alpha\overline{\beta}}(p, z) = \left(\frac{\partial G_\alpha}{\partial \overline{p}_\beta} + \frac{\partial G_\alpha}{\partial \overline{z}_\beta} \right) (p, z) \quad \text{for } (p, z) \in D \times D, 1 \leq \alpha, \beta \leq n,$$

and

$$(2.20) \quad g_{\alpha\overline{\beta}}(p, w) = \psi(p) \frac{\partial g_\alpha}{\partial \overline{p}_\beta}(p, w) - (2n-1)\psi_\beta(p)g_\alpha(p, w) - \psi_\beta(p) \sum_{i=1}^n \left(w_i \frac{\partial g_\alpha}{\partial w_i}(p, w) + \overline{w}_i \frac{\partial g_\alpha}{\partial \overline{w}_i}(p, w) \right),$$

$$p \in \overline{D}, w \in D(p), 1 \leq \alpha, \beta \leq n.$$

Then (2.19) can be written as

$$g_{\alpha\overline{\beta}}(p, w) = (\psi(p))^{2n} G_{\alpha\overline{\beta}}(p, z),$$

where $p, z \in D$ and $w = (z - p)/(-\psi(p))$. Differentiating the above relation with respect to z_c , we obtain

$$(2.21) \quad \frac{1}{-\psi} \frac{\partial g_{\alpha\overline{\beta}}}{\partial w_c} = \psi^{2n} \frac{\partial G_{\alpha\overline{\beta}}}{\partial z_c}.$$

On the otherhand, by [7, 6.14],

$$(2.22) \quad \frac{\partial^3 \Lambda}{\partial p_\alpha \partial \overline{p}_\beta \partial p_c}(p) = 2 \frac{\partial G_{\alpha\overline{\beta}}}{\partial z_c}(p, p).$$

Combining (2.21), (2.22) with (2.20), we obtain

$$(2.23) \quad \frac{\partial^3 \Lambda}{\partial p_\alpha \partial \overline{p}_\beta \partial p_c}(p)(\psi(p))^{2n} = -\frac{2}{\psi(p)} \frac{\partial g_{\alpha\overline{\beta}}}{\partial w_c}(p, 0) = -2 \frac{\partial^2 g_\alpha}{\partial w_c \partial \overline{p}_\beta}(p, 0) + \frac{4n}{\psi(p)} \psi_\beta(p) \frac{\partial g_\alpha}{\partial w_c}(p, 0).$$

Thus, information about the third order derivatives of Λ can be obtained by studying the derivatives of $g_\alpha(p, w)$.

Lemma 2.11. *Under the hypothesis of Theorem 2.4, we have for all $\alpha \in I$, $c \in I \cup \overline{I}$,*

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu\alpha}}{\partial w_c}(p_\nu, 0) = \frac{\partial g_\alpha}{\partial w_c}(0, 0) = (n-1)(2n-1)\psi_\alpha(p_0)\psi_c(p_0)|\partial\psi(p_0)|^{2n-2},$$

where $g_{\nu\alpha}(p, w)$ is defined by (2.17).

Proof. We know that $g_{\nu\alpha}(p_\nu, w)$ is a harmonic function of $w \in D_\nu(p_\nu)$ and $g_\alpha(p_0, w)$ is a harmonic function of $w \in \mathcal{H} = D(p_0)$. First, we will show that $\{g_{\nu\alpha}(p_\nu, w)\}$ converges uniformly on compact subsets of \mathcal{H} to $g_\alpha(p_0, w)$. The first equality then follows from the harmonicity of these functions. Note that $\{D_\nu(p_\nu)\}$ is a C^∞ -perturbation of \mathcal{H} (see the proof of Theorem 1.3 in [1]). Therefore by [1, Prop 3.1], $\{g_\nu(p_\nu, w)\}$ converges uniformly on compact subsets of $\mathcal{H} \setminus \{0\}$ to $g(p_0, w)$. By harmonicity,

$$\left\{ \frac{\partial g_\nu}{\partial w_i}(p_\nu, w) \right\}, \quad 1 \leq i \leq n,$$

also converges uniformly on compact subsets of $\mathcal{H} \setminus \{0\}$ to $(\partial g / \partial w_i)(p_0, w)$. Also by [1, Remark 4.5],

$$\left\{ \frac{\partial g_\nu}{\partial p_\alpha}(p_\nu, w) \right\}$$

converges uniformly on compact subsets of \mathcal{H} to $(\partial g / \partial p_\alpha)(p_0, w)$. It follows from (2.17) that $\{g_{\nu_\alpha}(p_\nu, w)\}$ converges uniformly to $g_\alpha(p_0, w)$ on compact subsets of $\mathcal{H} \setminus \{0\}$ and hence of \mathcal{H} by the mean value theorem.

Now to calculate $\frac{\partial g_\alpha}{\partial w_c}(p_0, 0)$ explicitly, let us write (2.17) in the form

$$(2.24) \quad g_\alpha(p, w) = \psi(p) \frac{\partial g}{\partial p_\alpha}(p, w) - (n-1) \psi_\alpha(p) (g_0(p, w) + \overline{g_0(p, w)}),$$

where

$$(2.25) \quad g_0(p, w) = g(p, w) + \frac{1}{n-1} \sum_{i=1}^n w_i \frac{\partial g}{\partial w_i}(p, w).$$

Note that if $w_0 \in \mathbf{C}^n$ then for $w \neq w_0$,

$$(2.26) \quad \sum_{i=1}^n w_i \frac{\partial}{\partial w_i} |w - w_0|^{-2n+2} = -(n-1) |w - w_0|^{-2n} \sum_{i=1}^n w_i \overline{(w_i - w_{0i})}.$$

This implies that the singularities in the right hand side of (2.25) get cancelled and hence (2.25) defines a harmonic function of $w \in D(p)$ for each $p \in \overline{D}$. From (2.6), we have

$$g(p_0, w) = |w|^{-2n+2} - |w - 0^*|^{-2n+2},$$

where $0^* = \overline{\partial \psi(p_0)} / |\partial \psi(p_0)|^2$ is the symmetric point of the origin with respect to $\partial D(p_0)$. Therefore,

$$\begin{aligned} (2.27) \quad g_0(p_0, w) &= -|w - 0^*|^{-2n+2} + |w - 0^*|^{-2n} \sum_{i=1}^n w_i \overline{(w_i - 0_i^*)} \\ &= -|w - 0^*|^{-2n} \left\{ |w - 0^*|^2 - \sum_{i=1}^n w_i \overline{(w_i - 0_i^*)} \right\} = |w - 0^*|^{-2n} \sum_{i=1}^n 0_i^* \overline{(w_i - 0_i^*)} \\ &= |w - 0^*|^{-2n} \left(-|0^*|^2 + \sum_{i=1}^n 0_i^* \overline{w_i} \right). \end{aligned}$$

From this equation we obtain

$$(2.28) \quad \frac{\partial}{\partial w_c} \{g_0(p_0, w) + \overline{g_0(p_0, w)}\} \Big|_{w=0} = -(2n-1) 0_c^* |0^*|^{-2n} = -(2n-1) \psi_c(p_0) |\partial \psi(p_0)|^{2n-2}.$$

Finally,

$$\begin{aligned} (2.29) \quad g_\alpha(p_0, w) &= -(n-1) \psi_\alpha(p_0) \{g_0(p_0, w) + \overline{g_0(p_0, w)}\} \\ &= (n-1) \psi_\alpha(p_0) |w - 0^*|^{-2n} \left\{ 2|0^*|^2 - \sum_{i=1}^n (0_i^* \overline{w_i} + \overline{0_i^*} w_i) \right\}, \end{aligned}$$

and hence by (2.28),

$$(2.30) \quad \frac{\partial g_\alpha}{\partial w_c}(p_0, 0) = (n-1)(2n-1) \psi_\alpha(p_0) \psi_c(p_0) |\partial \psi(p_0)|^{2n-2}$$

as desired. \square

Next we calculate the second the second order derivatives of g_α . To simplify the calculations, we will consider only a special case which is required for the proof of them main theorem.

Lemma 2.12. *Under the normalisat (\dagger) , we have for all $\alpha, \beta \in I$, $\beta \neq n$ and for all $c \in I \cup \bar{I}$,*

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \frac{\partial^2 g_{\nu\alpha}}{\partial w_c \partial \bar{p}_\beta}(p_\nu, 0) &= \frac{\partial^2 g_\alpha}{\partial w_c \partial \bar{p}_\beta}(0, 0) \\ &= \begin{cases} (n-1)(2n-1)\psi_\alpha(0)\psi_{\bar{\beta}c}(0) & \text{if } c \neq n, \bar{n} \\ (n-1)(2n-1)\psi_\alpha(0) \left\{ \left(n + \frac{1}{2}\right) \psi_{\bar{\beta}c}(0) + \left(n - \frac{1}{2}\right) \psi_{\bar{\beta}\bar{c}}(0) \right\} & \\ +(2n-1)\psi_{\alpha\bar{\beta}}(0) & \text{if } c = n \text{ or } c = \bar{n}. \end{cases} \end{aligned}$$

Proof. Let

$$\mathcal{H} = D(0) = \{w \in \mathbf{C}^n : 2\Re w_n - 1 < 0\}.$$

We will show that $\partial g_{\nu\alpha}/\partial \bar{p}_\beta$ converges uniformly on compact subsets of \mathcal{H} to $\partial g_\alpha/\partial \bar{p}_\beta$. Differentiating (2.17) with respect to \bar{p}_β ,

$$(2.31) \quad \frac{\partial g_\alpha}{\partial \bar{p}_\beta} = \psi \frac{\partial^2 g}{\partial \bar{p}_\beta \partial p_\alpha} + \frac{\partial \psi}{\partial \bar{p}_\beta} \frac{\partial g}{\partial p_\alpha} - \frac{\partial \psi}{\partial p_\alpha} \left\{ (2n-2) \frac{\partial g}{\partial \bar{p}_\beta} + \sum_{i=1}^n \left(w_i \frac{\partial^2 g}{\partial \bar{p}_\beta \partial w_i} + \bar{w}_i \frac{\partial^2 g}{\partial \bar{p}_\beta \partial \bar{w}_i} \right) \right\} \\ - \frac{\partial^2 \psi}{\partial \bar{p}_\beta \partial p_\alpha} \left\{ (2n-2)g + \sum_{i=1}^n \left(w_i \frac{\partial g}{\partial w_i} + \bar{w}_i \frac{\partial g}{\partial \bar{w}_i} \right) \right\}.$$

By Remark 4.5 of [1], $\{\frac{\partial g_\nu}{\partial p_\alpha}(p_\nu, w)\}$ and similarly by the arguments of Section 5 of [1], $\{\frac{\partial^2 g_\nu}{\partial \bar{p}_\beta \partial p_\alpha}(p_\nu, w)\}$ converges uniformly on compact subsets of \mathcal{H} to $\frac{\partial^2 g}{\partial \bar{p}_\beta \partial p_\alpha}(0, w)$. By harmonicity $\{\frac{\partial^2 g}{\partial w_i \partial \bar{p}_\beta}(p_\nu, w)\}$ converges uniformly on compact subsets of \mathcal{H} to $\frac{\partial^2 g}{\partial w_i \partial \bar{p}_\beta}(0, w)$. As in the previous lemma, $\{g_\nu(p_\nu, w)\}$ and $\{\frac{\partial g_\nu}{\partial w_i}(p_\nu, w)\}$ converges uniformly on compact subsets of $\mathcal{H} \setminus \{0\}$ to $g(p_0, w)$ and $\frac{\partial g}{\partial w_i}(0, w)$ respectively. Hence $\{\frac{\partial g_{\nu\alpha}}{\partial \bar{p}_\beta}(p_\nu, w)\}$ converges uniformly to $\frac{\partial g_\alpha}{\partial \bar{p}_\beta}(0, w)$ on compact subsets of $\mathcal{H} \setminus \{0\}$ and hence of \mathcal{H} by the mean value theorem. The first equality is now a consequence of harmonicity of these functions.

To calculate

$$\frac{\partial^2 g_\alpha}{\partial w_c \partial \bar{p}_\beta}(p_0, 0),$$

note that from (2.31),

$$\begin{aligned} \frac{\partial g_\alpha}{\partial \bar{p}_\beta}(0, w) &= -\psi_\alpha(0) \left\{ (2n-2) \frac{\partial g}{\partial \bar{p}_\beta}(0, w) + \sum_{i=1}^n \left(w_i \frac{\partial^2 g}{\partial \bar{p}_\beta \partial w_i}(0, w) + \bar{w}_i \frac{\partial^2 g}{\partial \bar{p}_\beta \partial \bar{w}_i}(0, w) \right) \right\} \\ &\quad - \psi_{\alpha\bar{\beta}}(0) \left\{ g_0(0, w) + \overline{g_0(0, w)} \right\}. \end{aligned}$$

Differentiating this with respect to w_c and using (2.28),

$$(2.32) \quad \frac{\partial^2 g_\alpha}{\partial w_c \partial \bar{p}_\beta}(0, 0) = -(2n-1)\psi_\alpha(0) \frac{\partial^2 g}{\partial w_c \partial \bar{p}_\beta}(0, 0) + (2n-1)\psi_c(0)\psi_{\alpha\bar{\beta}}(0).$$

Now,

From the work in [7, Chapter 4],

$$\frac{\partial g}{\partial \bar{p}_\beta}(0, w) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial \mathcal{H}} \overline{k_1^{(\beta)}(0, \zeta)} |\partial_\zeta g(0, \zeta)| \frac{\partial g_w}{\partial n_\zeta}(0, \zeta) dS_\zeta,$$

where

$$k_1^{(\beta)}(0, \zeta) = \frac{\partial f}{\partial p_\beta}(0, \zeta) / |\partial_\zeta f(0, \zeta)| = \sum_{j=1}^n \left(\zeta_j \psi_{\beta j}(0) + \bar{\zeta}_j \psi_{\beta \bar{j}}(0) \right),$$

and $g_w(0, \zeta)$ is the Green function for \mathcal{H} with pole at w . From the explicit formula (2.6),

$$g_w(0, \zeta) = |\zeta - w|^{-2n+2} - |\zeta - w^*|^{-2n+2}, \quad \zeta, w \in \mathcal{H},$$

where

$$w^* = (w_1, \dots, w_{n-1}, (1 - \Re w_n) + i\Im w_n)$$

is the symmetric point of w with respect to the hyperplane $\partial\mathcal{H}$. Therefore,

$$\frac{\partial g_w}{\partial \zeta_i}(0, \zeta) = -(n-1)\{|\zeta - w|^{-2n}(\bar{\zeta}_i - \bar{w}_i) - |\zeta - w^*|^{-2n}(\bar{\zeta}_i - \bar{w}_i^*)\}, \quad \zeta, w \in \mathcal{H}, 1 \leq i \leq n.$$

In particular, for $\zeta \in \partial H$, since $|\zeta - w| = |\zeta - w^*|$,

$$\frac{\partial g_w}{\partial \zeta_i}(0, \zeta) = -(n-1)|\zeta - w|^{-2n}\bar{w}_i^*, \quad w \in \mathcal{H}, 1 \leq i \leq n.$$

This implies that

$$|\partial_\zeta g(0, \zeta)| = (n-1)|\zeta|^{-2n}, \quad \zeta \in \partial H,$$

and

$$\frac{\partial g_w}{\partial n_\zeta}(0, \zeta) = \frac{\partial g_w}{\partial x_n}(0, \zeta) = -2(n-1)|\zeta - w|^{-2n}(1 - \Re w_n), \quad \zeta \in \partial H, w \in \mathcal{H}.$$

Therefore,

$$\begin{aligned} \frac{\partial g}{\partial \bar{p}_\beta}(0, w) = & -\frac{(n-1)(1 - \Re w_n)}{\sigma_{2n}} \sum_{j=1}^n \left\{ \psi_{\bar{\beta}j}(0) \int_{\partial\mathcal{H}} \zeta_j |\zeta|^{-2n} |\zeta - w|^{-2n} dS_\zeta \right. \\ & \left. + \psi_{\bar{\beta}j}(0) \int_{\partial\mathcal{H}} \bar{\zeta}_j |\zeta|^{-2n} |\zeta - w|^{-2n} dS_\zeta \right\}. \end{aligned}$$

Differentiating with respect to w_c ,

(2.33)

$$\begin{aligned} \frac{\partial^2 g}{\partial w_c \partial \bar{p}_\beta}(0, 0) = & -\frac{n(n-1)}{\sigma_{2n}} \sum_{j=1}^n \left\{ \psi_{\bar{\beta}j}(0) \int_{\partial\mathcal{H}} \zeta_j \bar{\zeta}_c |\zeta|^{-4n-2} dS_\zeta + \psi_{\bar{\beta}j}(0) \int_{\partial\mathcal{H}} \bar{\zeta}_j \bar{\zeta}_c |\zeta|^{-4n-2} dS_\zeta \right\} \\ & + \frac{(n-1)}{\sigma_{2n}} \frac{\partial \Re w_n}{\partial w_c} \sum_{j=1}^n \left\{ \psi_{\bar{\beta}j}(0) \int_{\partial\mathcal{H}} \zeta_j |\zeta|^{-4n} dS_\zeta + \psi_{\bar{\beta}j}(0) \int_{\partial\mathcal{H}} \bar{\zeta}_j |\zeta|^{-4n} dS_\zeta \right\} \end{aligned}$$

We now consider two cases:

Case I. $c \neq n, \bar{n}$. Let $1 \leq j \leq n$, and $1 \leq k \leq (n-1)$. Then integrating with respect to x_k and y_k variables first,

$$\int_{\partial\mathcal{H}} \zeta_j \zeta_k |\zeta|^{-4n-2} dS_\zeta = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{(x_j x_k - y_j y_k) + i(x_j y_k + y_j x_k)}{(x_1^2 + y_1^2 + \dots + 1/4 + x_{2n}^2)^{2n+1}} dx_1 dy_1 \dots \widehat{dx_n} dy_n = 0,$$

and also for $j \neq k$,

$$\int_{\partial\mathcal{H}} \zeta_j \bar{\zeta}_k |\zeta|^{-4n-2} dS_\zeta = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{(x_j x_k + y_j y_k) + i(y_j x_k - x_j y_k)}{(x_1^2 + y_1^2 + \dots + 1/4 + x_{2n}^2)^{2n+1}} dx_1 dy_1 \dots \widehat{dx_n} dy_n = 0.$$

It follows from (2.33) that

$$(2.34) \quad \frac{\partial^2 g}{\partial w_c \partial \bar{p}_\beta}(0, 0) = -\frac{n(n-1)}{\sigma_{2n}} \psi_{\bar{\beta}c}(0) \int_{\partial\mathcal{H}} |\zeta_c|^2 |\zeta|^{-4n-2} dS_\zeta.$$

Now, if $1 \leq k \leq n-1$,

(2.35)

$$\begin{aligned} \frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} |\zeta_k|^2 |\zeta|^{-4n-2} dS_\zeta &= \frac{1}{\sigma_{2n}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{x_k^2 + y_k^2}{(x_1^2 + y_1^2 + \dots + 1/4 + x_{2n}^2)^{2n+1}} dx_1 dy_1 \dots \widehat{dx_n} dy_n \\ &= \frac{2}{\sigma_{2n}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{x_k^2}{(x_1^2 + y_1^2 + \dots + 1/4 + y_n^2)^{2n+1}} dx_1 dy_1 \dots \widehat{dx_n} dy_n \equiv 2A. \end{aligned}$$

Note that for $K > 0$ and $m > 1$, integrating by parts,

$$\int_{-\infty}^{\infty} \frac{t^2}{(t^2 + K)^{m+1}} dt = \frac{1}{2m} \int_{-\infty}^{\infty} \frac{1}{(t^2 + K)^m} dt.$$

Therefore, taking $m = 2n$, $K = x_1^2 + y_1^2 + \cdots + \widehat{x_i^2} + y_i^2 + \cdots + 1/4 + y_n^2$,

$$(2.36) \quad A = \frac{1}{4n\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(x_1^2 + y_1^2 + \cdots + 1/4 + y_n^2)^{2n}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n = \frac{1}{4n} X = \frac{1}{2n}$$

where X is as in lemm 2.6. Hence from (2.34) and (2.35),

$$\frac{\partial^2 g}{\partial w_c \partial \overline{p}_\beta}(0) = -(n-1)\psi_{\overline{\beta}c}(0).$$

Therefore, from (2.32),

$$\frac{\partial^2 g_\alpha}{\partial w_c \partial \overline{p}_\beta}(0,0) = (n-1)(2n-1)\psi_\alpha(0)\psi_{\overline{\beta}c}(0) \quad \text{if } c \neq n, \overline{n}.$$

Case II. $c = n$ or $c = \overline{n}$. Let $1 \leq j \leq n-1$. Then integrating with respect to x_j and y_j variables first,

$$\int_{\partial\mathcal{H}} \zeta_j \zeta_n |\zeta|^{-4n-2} dS_\zeta = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{(x_j/2 - y_j y_n) + i(x_j y_n + y_j/2)}{(x_1^2 + y_1^2 + \cdots + 1/4 + x_{2n}^2)^{2n+1}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n = 0$$

and similarly

$$\int_{\partial\mathcal{H}} \zeta_j \overline{\zeta}_n |\zeta|^{-4n-2} dS_\zeta = 0, \quad \int_{\partial\mathcal{H}} \zeta_j |\zeta|^{-4n} dS_\zeta = 0.$$

It follows from (2.33) that

$$(2.37) \quad \frac{\partial^2 g}{\partial w_c \partial \overline{p}_\beta}(0) = -\frac{n(n-1)}{\sigma_{2n}} \left\{ \psi_{\overline{\beta}c}(0) \int_{\partial\mathcal{H}} |\zeta_n|^2 |\zeta|^{-4n-2} dS_\zeta + \psi_{\overline{\beta}\overline{c}}(0) \int_{\partial\mathcal{H}} \zeta_c^2 |\zeta|^{-4n-2} dS_\zeta \right\} \\ + \frac{(n-1)}{2\sigma_{2n}} \left\{ \psi_{\overline{\beta}n}(0) \int_{\partial\mathcal{H}} \zeta_n |\zeta|^{-4n} dS_\zeta + \psi_{\overline{\beta}\overline{n}}(0) \int_{\partial\mathcal{H}} \overline{\zeta}_n |\zeta|^{-4n} dS_\zeta \right\}.$$

Now,

$$\frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} |\zeta_n|^2 |\zeta|^{-4n-2} dS_\zeta = \frac{1}{4\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(x_1^2 + y_1^2 + \cdots + 1/4 + y_n^2)^{2n+1}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n \\ + \frac{1}{\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{y_n^2}{(x_1^2 + y_1^2 + \cdots + 1/4 + x_{2n}^2)^{2n+1}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n \equiv \frac{1}{4} B + A.$$

Then,

$$B = \frac{1}{\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(x_1^2 + y_1^2 + \cdots + 1/4 + y_n^2)^{2n+1}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n \\ = \frac{\sigma_{2n-1}}{\sigma_{2n}} \int_0^\infty \frac{r^{2n-2}}{(r^2 + 1/4)^{2n+1}} dr = \frac{\sigma_{2n-1}}{\sigma_{2n}} I(2n-2, 2n+1).$$

As in Lemma 2.6,

$$I(2n-2, 2n+1) = \frac{2n-3}{2(2n)} \frac{2n-5}{2(2n-1)} \cdots \frac{1}{2(n+2)} I(0, n+2),$$

and

$$I(0, n+2) = \frac{\pi}{(n+1)!} (n+2)(n+3) \cdots (2n+2),$$

so that

$$B = \left\{ \frac{1}{\pi} \frac{2^{n-1}(n-1)!}{(2n-3)(2n-5) \cdots 1} \right\} \left\{ \frac{2n-3}{2(2n)} \frac{2n-5}{2(2n-1)} \cdots \frac{1}{2(n+2)} \right\} \left\{ \frac{\pi}{(n+1)!} (n+2)(n+3) \cdots (2n+2) \right\} \\ = \frac{2(2n+1)}{n}.$$

Therefore,

$$\frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} |\zeta_n|^2 |\zeta|^{-4n-2} dS_\zeta = \frac{(n+1)}{n}.$$

Also,

$$\begin{aligned} \frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} \zeta_{\bar{c}}^2 |\zeta|^{-4n-2} dS_{\zeta} &= \frac{1}{\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{(1/4 - y_n^2) \pm iy_n}{(x_1^2 + y_1^2 + \cdots + 1/4 + x_{2n}^2)^{2n+1}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n \\ &= \frac{1}{4} B - A = 1, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\sigma_{2n}} \int_{\partial\mathcal{H}} \zeta_n |\zeta|^{-4n} dS_{\zeta} &= \frac{1}{\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1/2 + iy_n}{(x_1^2 + y_1^2 + \cdots + 1/4 + x_{2n}^2)^{2n}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n \\ &= \frac{1}{2} \frac{1}{\sigma_{2n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(x_1^2 + y_1^2 + \cdots + 1/4 + x_{2n}^2)^{2n}} dx_1 dy_1 \cdots \widehat{dx_n} dy_n = 2nA = 1. \end{aligned}$$

Hence from (2.37),

$$\frac{\partial^2 g}{\partial w_c \partial \bar{p}_{\beta}}(0, 0) = -(n-1) \left\{ \left(n + \frac{1}{2} \right) \psi_{\bar{\beta}c}(0) + \left(n - \frac{1}{2} \right) \psi_{\bar{\beta}\bar{c}}(0) \right\}.$$

Thus from (2.32),

$$\begin{aligned} \frac{\partial^2 g_{\alpha}}{\partial w_c \partial \bar{p}_{\beta}}(0, 0) &= (n-1)(2n-1) \psi_{\alpha}(0) \left\{ \left(n + \frac{1}{2} \right) \psi_{\bar{\beta}c}(0) + \left(n - \frac{1}{2} \right) \psi_{\bar{\beta}\bar{c}}(0) \right\} \\ &\quad + (2n-1) \psi_{\alpha\bar{\beta}}(0), \quad \text{if } c = n \text{ or } c = \bar{n}, \end{aligned}$$

as desired. \square

Combining the informations about the derivatives of $g_{\nu\alpha}$, we obtain the following asymptotics of the third derivatives of Λ_{ν} :

Proposition 2.13. *Under the normalisation (\dagger) , we have for all $\alpha, \beta, \gamma \in I$, $\beta \neq n$,*

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \frac{\partial^3 \Lambda_{\nu}}{\partial p_{\alpha} \partial \bar{p}_{\beta} \partial p_c} (p_{\nu}) \psi_{\nu}(p_{\nu})^{2n} \\ = \begin{cases} -2(n-1)(2n-1) \psi_{\alpha}(0) \psi_{\bar{\beta}c}(0) & \text{if } c \neq n, \bar{n}, \\ -(n-1)(2n-1) \psi_{\alpha}(0) \{ \psi_{\bar{\beta}c}(0) - \psi_{\bar{\beta}\bar{c}}(0) \} - 2(2n-1) \psi_{\alpha\bar{\beta}}(0) & \text{if } c = n \text{ or } c = \bar{n}. \end{cases} \end{aligned}$$

Proof. Consider the formula (2.23) corresponding to Λ_{ν} and apply Lemma 2.5, Lemma 2.11 and Lemma 2.12 to obtain the desired result. \square

We conclude this section with the following calculation:

Corollary 2.14. *Under the normalisation (\dagger) , we have for all $\alpha, \beta \in I$, $\beta \neq n$ and $c \in I \cup \bar{I}$,*

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_{\alpha\bar{\beta}}}{\partial p_c} (p_{\nu}) (\psi(p_{\nu}))^2$$

exists and is finite.

Proof. From (2.9),

$$\begin{aligned} \frac{\partial g_{\alpha\bar{\beta}}}{\partial p_c} \psi^2 &= \frac{\Lambda_{\alpha\bar{\beta}c} \psi^{2n}}{\Lambda \psi^{2n-2}} - \frac{(\Lambda_{\alpha\bar{\beta}} \psi^{2n-1})(\Lambda_c \psi^{2n-1}) + (\Lambda_{\alpha c} \psi^{2n})(\Lambda_{\bar{\beta}} \psi^{2n-2}) + (\Lambda_{\bar{\beta}c} \psi^{2n-1})(\Lambda_{\alpha} \psi^{2n-1})}{(\Lambda \psi^{2n-2})^2} \\ &\quad + \frac{2(\Lambda_{\alpha} \psi^{2n-1})(\Lambda_{\bar{\beta}} \psi^{2n-2})(\Lambda_c \psi^{2n-1})}{(\Lambda \psi^{2n-2})^3}. \end{aligned}$$

Frist let $c \neq n, \bar{n}$. Then applying Corollay 2.2, Corollary 2.7 and Proposition 2.13 to the above formula corresponding to Λ_{ν} , we obtain

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu\alpha\bar{\beta}}}{\partial p_c} (p_{\nu}) \psi(p_{\nu})^2 = 2(n-1)(2n-1) \psi_{\alpha}(0) \psi_{\bar{\beta}c}(0) - (2n-2)^2 \psi_{\alpha}(0) \psi_{\bar{\beta}c}(0) = 2(n-1) \psi_{\alpha}(0) \psi_{\bar{\beta}c}(0).$$

Similarly for $c = n$ or $c = \bar{n}$,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu \alpha \bar{\beta}}}{\partial p_c}(p_\nu) \psi(p_\nu)^2 &= (n-1)(2n-1) \psi_\alpha(0) \{ \psi_{\bar{\beta}c}(0) - \psi_{\bar{\beta}\bar{c}}(0) \} + 2(2n-1) \psi_{\alpha\bar{\beta}}(0) \\ &\quad + 4(n-1)^2 \psi_\alpha(0) \{ \psi_{\bar{\beta}n}(0) + \psi_{\bar{\beta}\bar{n}}(0) \} - 4(n-1)^2 \psi_{\alpha\bar{\beta}}(0) - 4(n-1)^2 \psi_\alpha(0) \psi_{\bar{\beta}c}(0) \\ &= (n-1) \psi_\alpha(0) \{ (2n-1) \psi_{\bar{\beta}c}(0) + (2n-3) \psi_{\bar{\beta}\bar{c}}(0) \} + 2\{ (2n-1) - 2(n-1)^2 \} \psi_{\alpha\bar{\beta}}(0). \end{aligned}$$

□

3. GEODESIC SPIRALS : PROOF OF PROPOSITION 1.2

Proof. We prove this proposition by contradiction. Suppose the assertion is not true. Then there exists a sequence $\{c_\nu\}$ of geodesics with the following properties:

- (i) There exists a point $a_0 \in \partial D$ such that $a_\nu := c_\nu(0)$ converges to a_0 as $\nu \rightarrow \infty$.
- (ii) The unit vectors $u_\nu := \frac{c'_\nu(0)}{|c'_\nu(0)|}$ converges to a unit vector u_0 .
- (iii) We have $(\psi \circ c_\nu)'(0) = 0$ and $(\psi \circ c_\nu)''(0) \leq 0$ for each ν .

Since the Λ -metric is invariant under affine transformations, without loss of generality let us assume that

- $a_0 = 0$, $\partial\psi(0) = (0, \dots, 0, 1)$, and $v_0 = (1, 0, \dots, 0)$.

If ν is sufficiently large, then the distance between a_ν and ∂D , say δ_ν , is realised by a unique point $\pi(a_\nu) \in \partial D$, i.e.,

$$\delta_\nu = d(a_\nu, \partial D) = |a_\nu - \pi(a_\nu)|.$$

We again assume without loss of generality that this is true for all $\nu \geq 1$. Now for each ν , we apply a translation followed by sufficiently many rotations to transform the domain D to a new domain D_ν with a global defining function ψ_ν , such that

- $\pi(a_\nu) \in \partial D$ corresponds to $0 \in \partial D_\nu$ and $\partial\psi_\nu(0) = (0, \dots, 0, 1)$.
- The geodesic c_ν in D corresponds to the geodesic γ_ν in D_ν that has the following properties:
 - (a) $p_\nu := \gamma_\nu(0) = (0, \dots, 0, -\delta_\nu)$.
 - (b) $v_\nu := \frac{\gamma'_\nu(0)}{|\gamma'_\nu(0)|} = (1, 0, \dots, 0)$.
 - (c) $(\psi_\nu \circ \gamma_\nu)'' \leq 0$.

Note that the above three bullets imply D and D_ν are as in the normalisation (†). In what follows we will derive a contradiction by showing that

$$\lim_{\nu \rightarrow \infty} \frac{(\psi_\nu \circ \gamma_\nu)''(0)}{|\gamma'_\nu(0)|^2} > 0.$$

We start with the following lemma:

Lemma 3.1. *If $\gamma = (\gamma_1, \dots, \gamma_n)$ is a geodesic of the Λ -metric on D , then*

$$(\psi \circ \gamma)'' = -2\Re \sum_{\alpha=1}^n \psi_\alpha(\gamma) \sum_{j,k=1}^n \left(\sum_{\beta=1}^n \frac{\partial g_{k\bar{\beta}}}{\partial p_j} g^{\beta\bar{\alpha}} \right) (\gamma) \gamma'_j \gamma'_k + 2\Re \sum_{j,k=1}^n \psi_{\alpha\beta}(\gamma) \gamma'_\alpha \gamma'_\beta + 2\mathcal{L}_\psi(\gamma, \gamma').$$

Proof. Note that

$$(\psi \circ \gamma)'' = 2\Re \sum_{\alpha=1}^n \psi_\alpha(\gamma) \gamma''_\alpha + 2\Re \sum_{\alpha,\beta=1}^n \psi_{\alpha\beta}(\gamma) \gamma'_\alpha \gamma'_\beta + 2\mathcal{L}_\psi(\gamma, \gamma').$$

On the other hand, the equations of geodesic in the complexified form is given by

$$-\gamma''_\alpha = \sum_{j,k=1}^n \left(\sum_{\beta=1}^n \frac{\partial g_{k\bar{\beta}}}{\partial p_j} g^{\beta\bar{\alpha}} \right) (\gamma) \gamma'_j \gamma'_k.$$

Substituting this in the above formula yields the lemma. □

Now, this lemma together with (b) implies that

$$(3.1) \quad \frac{(\psi_\nu \circ \gamma_\nu)''(0)}{|\dot{\gamma}_\nu(0)|^2} = -2\Re \left(\sum_{\alpha, \beta=1}^n \frac{\partial \psi_\nu}{\partial p_\alpha} \frac{\partial g_{1\bar{\beta}}}{\partial p_1} g^{\beta\bar{\alpha}} \right) (p_\nu) + 2\Re \frac{\partial^2 \psi_\nu}{\partial p_1^2} (p_\nu) + 2 \frac{\partial^2 \psi_\nu}{\partial p_1 \partial \bar{p}_1} (p_\nu) \\ \equiv -2\Re I + 2\Re II + 2II.$$

We will now compute the limit of I as $\nu \rightarrow \infty$. For convenience, we will drop the subscript ν . We write I as

$$I = \sum_{\alpha=1}^{n-1} \sum_{\beta=1}^n \psi_\alpha \frac{\partial g_{1\bar{\beta}}}{\partial p_1} g^{\beta\bar{\alpha}} + \sum_{\beta=1}^{n-1} \psi_n \frac{\partial g_{1\bar{\beta}}}{\partial p_1} g^{\beta\bar{n}} + \psi_n \frac{\partial g_{1\bar{n}}}{\partial p_1} g^{n\bar{n}} \equiv A + B + C.$$

Claim: $A \rightarrow 0$ as $\nu \rightarrow \infty$. Write

$$A = \sum_{\alpha=1}^{n-1} \sum_{\beta=1}^n \left(\frac{\psi_\alpha}{\psi} \right) \left(\frac{\partial g_{1\bar{\beta}}}{\partial p_1} \psi^3 \right) \left(\frac{g^{\beta\bar{\alpha}}}{\psi^2} \right).$$

As $\nu \rightarrow \infty$, the first bracket converges by lemma 2.5, the second one converges to 0 by corollary 2.3 and third one converges by corollary 2.10. Thus $A \rightarrow 0$ as $\nu \rightarrow \infty$.

Claim: $B \rightarrow 0$ as $\nu \rightarrow \infty$. Write

$$B = \sum_{\beta=1}^{n-1} \psi_n \left(\frac{\partial g_{1\bar{\beta}}}{\partial p_1} \psi^2 \right) \frac{g^{\beta\bar{n}}}{\psi^2}$$

As $\nu \rightarrow \infty$, the first bracket converges to 1, the second one converges to 0 by corollary 2.14 and third one converges by corollary 2.10. Thus $B \rightarrow 0$ as $\nu \rightarrow \infty$.

Claim: $C \rightarrow \psi_{11}(0)$ as $\nu \rightarrow \infty$. Write

$$C = \psi_n \left(\frac{\partial g_{1\bar{n}}}{\partial p_1} \psi^2 \right) \left(\frac{g^{n\bar{n}}}{\psi^2} \right).$$

As $\nu \rightarrow \infty$, $\frac{\partial \psi}{\partial p_n} \rightarrow 1$ and by Corollary 2.10,

$$\frac{g^{n\bar{n}}}{\psi^2} \rightarrow \frac{1}{2(n-1)}.$$

Also by Corollary 2.14,

$$\frac{\partial g_{1\bar{n}}}{\partial p_1} = \overline{\frac{\partial g_{n\bar{1}}}{\partial \bar{p}_1}} \rightarrow \overline{2(n-1)\psi_{1\bar{1}}(0)} = 2(n-1)\psi_{11}(0).$$

Thus $C \rightarrow \psi_{11}(0)$ as $\nu \rightarrow \infty$.

It follows that $I \rightarrow \psi_{11}(0)$ as $\nu \rightarrow \infty$. Evidently $II \rightarrow \psi_{11}(0)$ as $\nu \rightarrow \infty$. Hence from (3.1),

$$\lim_{\nu \rightarrow \infty} \frac{(\psi_\nu \circ \gamma_\nu)''(0)}{|\dot{\gamma}_\nu(0)|^2} = 2\psi_{1\bar{1}}(0) > 0$$

as D is strongly pseudoconvex. This contradicts (c) and hence the proposition is proved. \square

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